Précis of *Making Sense of Number, Bit-by-Bit*

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In 1871, the political economist and logician William Stanley Jevons published a paper in Nature describing the results of a bean-counting experiment in which he was the sole participant. His goal was to determine "how many objects the mind can apprehend at once," which he tested by repeatedly tossing beans into a bucket and estimating their quantity. His experiment revealed several puzzling behavioral patterns that have since been replicated innumerable times. First, there is a sharp cut-off at about four objects, now called the "subitizing range," below which estimation is error-free. Beyond that cutoff, imprecision in estimation grows with magnitude — specifically, the standard deviation of estimates grows almost exactly linearly. Larger quantities are also underestimated, with the degree of underestimation increasing as a function of quantity. Data from Jevons' experiment have been re-plotted and are shown in Figure 1.

Jevons' results were published 150 years ago, but the study of innate numerical capacities has retained substantial interest in recent decades (Dehaene, 2011). There are several reasons for this, including: that basic numerical abilities seem to be present even in simple animals and are observed early in human development; that numerosity seems to be a primary perceptual attribute (Burr & Ross, 2008; Ross & Burr, 2010), susceptible to adaptation; that number psychophysics are unique among visual summary statistics in their discontinuity below and above four objects (Revkin et al., 2008); and that innate numerical abilities have been central to developmental theories of counting acquisition and other learned numerical abilities (Carey, 2009; Gelman & Gallistel, 1978; Wynn, 1992). The literature on numerical cognition in general, and visual numerosity perception in particular, is therefore quite substantial — and the psychophysics of numerosity perception are accordingly well-characterized.

This thesis contains experiments, models, and analyses that are centered on understanding the functional and mechanistic origins of observed number psychophysics. The findings presented here challenge some widely held assumptions and offer new ways of understanding some of the most puzzling phenomena. For instance, our results indicate that estimation is not at all a static and parallel process, as is commonly believed, but is actually the result of a serial accumulation process operating over saccades. We also present evidence that the discontinuity in estimation below and above four objects — generally held to be the result of two separate representational systems (Carey, 2009; Feigenson et al., 2004) — is in fact an efficient representation of quantity in a single system. Finally, our results show that subitizing, Weber's law, underestimation, and other key psychophysics are not due to properties of a *number system* (or systems), but rather emerge from lower-level perceptual uncertainty.

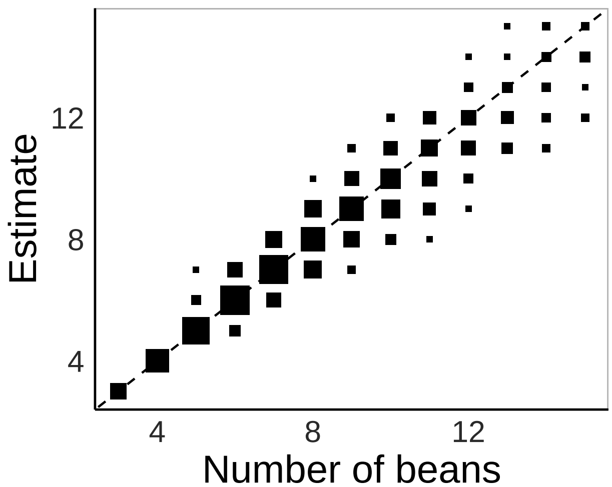

Figure 1: Data from Jevons (1871), showing the distribution of his estimates (y-axis) for each actual quantity of beans thrown (x-axis), with the size of the square proportional to how many times that number-estimate pair occurred.

The visual mechanics of estimation

Approximate numerical estimation is widely held to be a static, parallel, and nearly immediate process (Anobile et al., 2014; Dehaene, 1997; Revkin et al., 2008) — assumptions baked into nearly all computational models of numerosity perception (e.g. Dehaene & Changeux, 1993; Stoianov & Zorzi, 2012; Testolin et al., 2020; Verguts & Fias, 2004; Zorzi & Testolin, 2018). This view is supported by response times: whereas counting takes around 300ms per enumerated item, approximate number computations can take as little as 16ms independent of the number of objects (Inglis & Gilmore, 2013). Additionally, populations of neurons have been identified that respond similarly to sequentially and simultaneously-presented numerosities in monkeys (Nieder et al., 2006), which has been taken as evidence that approximate number representations are not the result of sequential processing.

However, recent evidence has muddied the simple picture of numerical estimation. Several studies have shown that individuals' Weber fractions are highly task-dependent, differing between estimation and discrimination tasks (e.g. Guillaume & Gevers, 2016; Price et al., 2012). In fact, Weber fractions have poor re-test reliability even when measured using the same task (Inglis & Gilmore, 2014). Numerical estimates have also been found to be influenced by non-numerical features of stimuli, such as the degree of clustering in a scene (Im et al., 2016). Finally, the precision of numerical estimates is known to improve as stimuli are presented for a longer duration (Inglis & Gilmore, 2013), suggesting that estimation may involve some type of temporal process.

Chapter 2 is an attempt to understand these temporal dynamics and the visual mechanics that underlie them. We ran estimation and discrimination tasks in which participants made non-symbolic numerosity judgments at different exposure durations and, critically, collected visual fixation data using an eye-tracker so that we could measure how participants' estimates were influenced by their path of visual fixations. We found that both mean estimates and the precision of estimates are strongly influenced by time, such that participants are biased to underestimate at short exposure durations and highly imprecise, but become increasingly un-biased and precise at longer exposure times, indicative of sequential (non-parallel) processing.

However, the effects of time appear to be almost entirely driven by visual fixations: as people

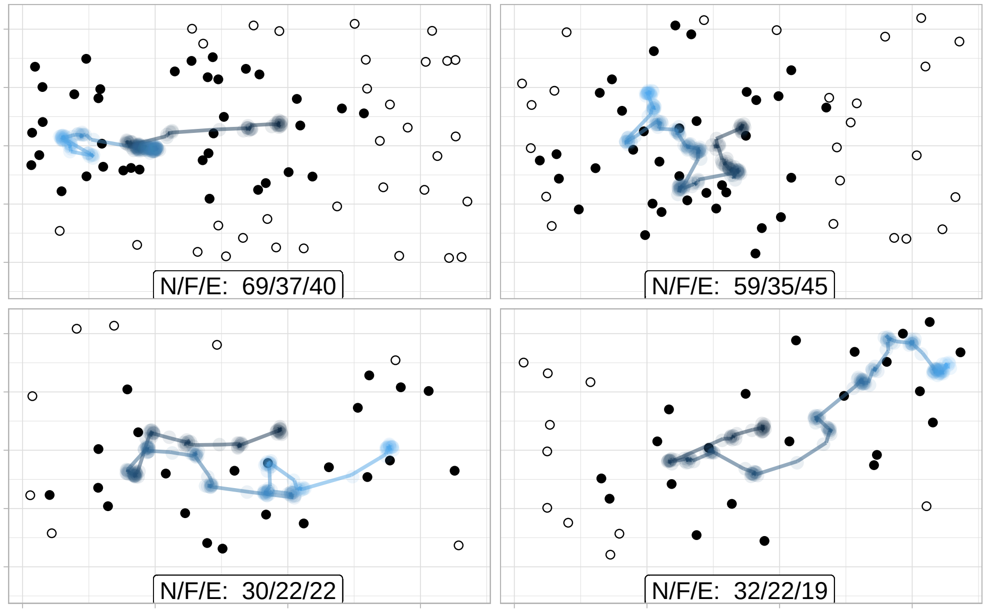

Figure 2: Example fixation paths of one subject in the 3-second time condition, with each panel representing a single trial. The points represent the dots displayed on their screen, where filled dots represent the ones that were foveated. At the bottom of each panel, a label N/F/E shows how many dots were shown (N), how many were foveated (F) and what quantity the participant actually estimated (E).

foveate an increasing number of objects, their mean estimates are driven up and their uncertainty is driven down. Figure 2 shows four example trials with participants' fixation paths across a scene containing multiple objects. The bottom of each panel shows the total number of points on the screen, the number of points that participants fixated, and the number they they ultimately estimated. There is a striking correspondence between how many points people fixated and their estimates, as if people are simply using an approximate count of the objects they happened to foveate to guide their estimate — without even adjusting for the area they did not fixate.

We used a statistical model to quantify the contribution of foveal, peripheral, and multiply-fixated dots in an array that, perhaps surprisingly, supports this interpretation. Freely fit parameters from our model indicate that foveated points contribute twice as much to a numerical estimate as peripheral ones. The analysis also revealed that estimates do not seem to be adjusted for the area of the screen that participants happened to fixate. Our results therefore suggest that estimation seems to be largely a simple process of serial accumulation of quantity across saccades. Our findings challenge the standard view of estimation as a static, pre-attentive, and parallel visual process, and imply that models of estimation that do not account for temporal dynamics — such as those based on feedforward neural networks (e.g. Dehaene & Changeux, 1993; Stoianov & Zorzi, 2012; Testolin et al., 2020; Verguts & Fias, 2004; Zorzi & Testolin, 2018) — will be unable to account for a large portion of the variance in estimation.

A unifying account of small and large number perception

Kaufman et al. (1949) delineated three distinct modes of enumeration: counting, subitizing, and estimating. Counting, of course, is a learned, serial procedure for exactly determining how many objects are in a set of arbitary size. Subitizing and estimation, on the other hand, are distinguished from counting as innate processes of enumeration that can operate in parallel over a visual field.

Subitizing, a term they coined from the Latin word *subitare* meaning "to arrive suddenly," is the fast and accurate mode of apprehending small quantities. They considered the quantity of sets with six or fewer members to be "subitized" rather than "estimated." Estimation, on the other hand, is a less accurate and somewhat slower mode of determining the numerosity of larger sets. They speculated, based on data regarding participants' reaction times, accuracy, and confidence in quantity estimation, that there is some mechanistic distinction between subitizing and estimation. They wrote,

> The two terms differ in meaning, because to produce the process of *estimating* we present more than 6 dots; to produce *subitizing* we present 6 or less. This difference is surely an identifiable difference in operations. It might be a trivial difference, but the results tell us that it is not. If no discontinuities had appeared in the results, no distinction between subitizing and estimating could have been drawn.

Though the subitizing range is now considered to be four rather than six, the idea that there are two operational modes of determining a set's quantity (other than counting) is widely accepted. Furthermore, Kaufman et al.'s suggestion that these modes reflect different underlying cognitive mechanisms has gained widespread support as well. On one prominent account, we have two innate systems that allow us to represent numerical information (Dehaene, 1997; Feigenson et al., 2004; Trick & Pylyshyn, 1994). The first is the "parallel individuation" system, which allows us to attend to and track up to four objects. This slot-like tracking mechanism is what allows for rapid, exact enumeration of small quantities. The second is the "approximate number system," which is a noisy, analog system for representing numerosity in sets when their size exceeds the limits of the parallel individuation system.

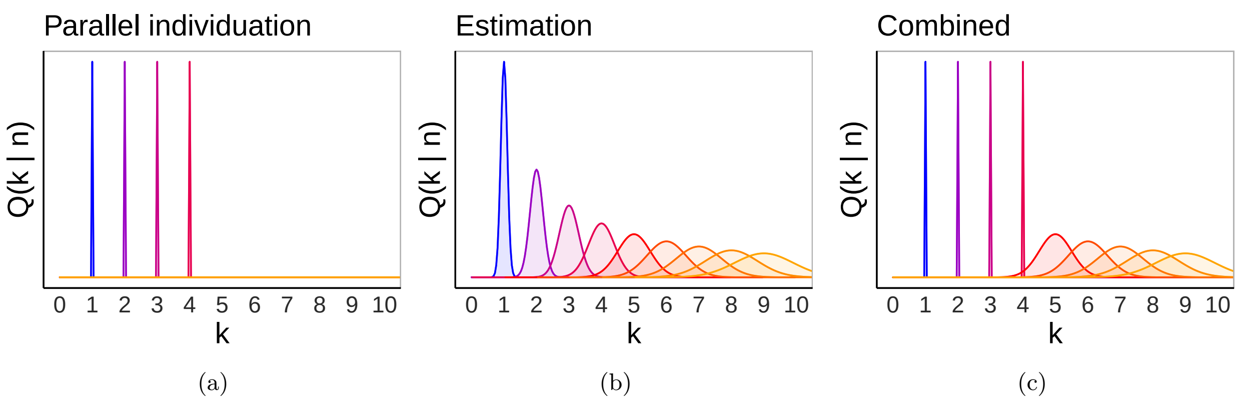

Figure 3: Distributions $Q(k|n)$ of responses (k) given a number of objects presented (n) under three models. Probabilities (y-axis) of estimates (x-axis) are shown for numerosities 1-9 (colors). Panel (a) shows the form of a precise estimation system, panel (b) shows the form of a scale variable estimation system, and panel (c) shows them combined.

The psychophysics of estimation resulting from the parallel individuation system and the approximate number system are illustrated in Figure 3. Each line represents the probability density $Q(k \mid n)$ over estimates (k) given a number of objects presented (n). Given $n = 1...4$ objects, the parallel individuation system exactly tracks them and thus estimation will be perfectly accurate, illustrated by the delta function probability density curves in Figure 3a. However, it is unable to represent sets beyond $n = 4$. The approximate number system (exemplified by 3b), on the other hand, is analog, continuous, and unbounded but represents each subsequent numerosity with decreasing precision. A popular model of the approximate number system assumes that estimates are

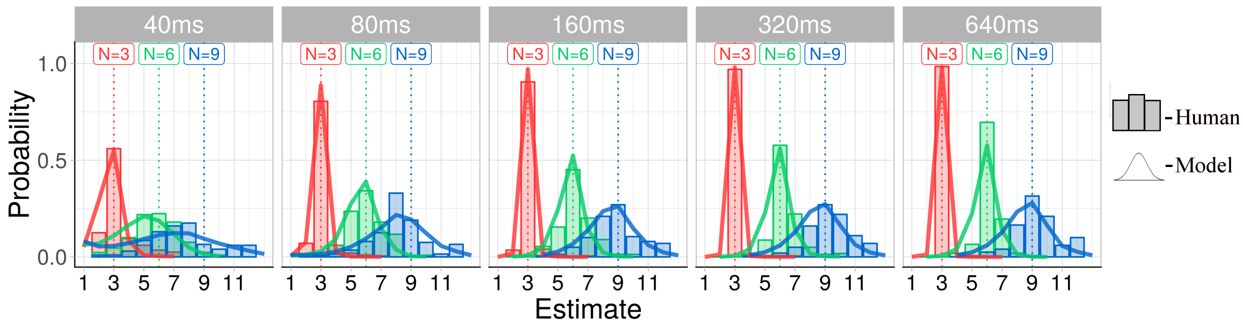

Figure 4: The probability (y-axis) of numeric responses (x-axis) over presentation times (faceted) for N=3, N=6, and N=9. Bars are shown for the human data and lines are shown for the model predictions.

drawn from, $k \sim \mathcal{N}(n, w \cdot n)$. This model of large number estimation has the standard deviation of estimates increasing at a rate w per object shown, where the constant w is called a person's "Weber fraction."

Chapter 3 addresses the origins of small- and large-number psychophysics, centering on the curious discontinuity in estimation error between four and five objects[1]. We take as a starting point the natural "need probability" (Anderson & Schooler, 1991) of number, which robustly follows a $1/n^2$ power-law in both number words (Dehaene & Mehler, 1992; Piantadosi & Jacobs, 2016) and how often numerosities are encountered and used for decision-making in the wild (Piantadosi & Cantlon, 2017; Strandburg-Peshkin et al., 2015). Using optimization methods from information theory, we derive the most efficient way to represent numerosities given a limited processing capacity (a fixed "informational budget"). Our derivation captures the core properties of number psychophysics, including (i) nearly exact representations for small sets (Burr et al., 2010; Choo & Franconeri, 2014; Feigenson et al., 2004; Revkin et al., 2008); (ii) scalar variability in estimation for larger numbers (Dehaene, 2011; Xu & Spelke, 2000); (iii) an underestimation bias (Izard & Dehaene, 2008; Mandler & Shebo, 1982) that diminishes with exposure time (Cheyette & Piantadosi, 2019); (iv) large number estimation acuity that is modulated by time (Inglis & Gilmore, 2013) and display contrast; (v) a subitizing range that is moderated by time (Mandler & Shebo, 1982) and contrast (Melcher & Piazza, 2011); and (vi) roughly normally-shaped response distributions for estimation (Nieder & Dehaene, 2009; Pica et al., 2004).

Beyond simply re-capitulating known psychophysics of number, the model makes testable predictions about how estimation acuity, subitizing range, and underestimation bias should depend on the amount of information available to participants. We evaluated these predictions against human behavior in four numerical estimation experiments, which reflect different ways of manipulating available information (variable exposure time versus display contrast) and different ways of controlling non-numerical properties of the stimuli (the average dot size, surface area, or density of the dots). We found the the model's predictions correspond closely to observed human number psychophysics in each experiment. For instance, at increasingly short exposure durations, participants' subitizing range is gradiently reduced — from four all the way down to about one at the shortest exposure times (or lowest color contrast) — matching a key prediction of the model.

Perhaps more importantly, the model closely matches the shape of the *distribution* of estimates under different conditions. Figure 4 shows the shape of the model (line) and human (bar) response

[1]This work was published in *Nature Human Behaviour*, as Cheyette and Piantadosi (2020), and can be found at https://www.nature.com/articles/s41562-020-00946-0.

distributions for $N = 3, 6, 9$ at different exposure durations (facets). These make it clear that it is not just the means and standard deviations which match closely, but rather the shape of the entire distribution derived from the optimization.

In sum, the theory we present relies on combining an a priori biological consideration (bounded informational capacity) with an environmental input distribution $P(n)$ and analytically computing the optimal internal representation. The resulting representational system replicates all of the standard properties of number psychophysics and explains them with a simple, resource-rational model. Our experiments also show that human numerical cognition quantitatively tracks this bounded optimal solution as the amount of information available varies, a fact not explainable in existing psychophysical theories. This work highlights that behavioral discontinuities are not always good markers of distinct systems: the fact that a single optimization produces the discontinuous psychophysics of number estimation implies that observing a discontinuity cannot be used as evidence of two representational systems.

The *perceptual* origins of number psychophysics

A key unresolved question is whether the behavioral patterns found in the domain of number result from numerical processing itself or from some of the perceptual processes that feed into numerical perception. In the first case, people may posses a "number system" that itself is the origin of phenomena seen in behavioral tasks involving number, such as Weber's law and underestimation. For instance, the noise and bias observed in numerical estimation might arise from a sampling process in which numerical information is extracted from visual representations, rather than from noise inherent to visual representations themselves (Dehaene & Changeux, 1993; Heng et al., 2020; Woodford, 2020). Alternatively, such phenomena may emerge as a consequence of more general visual processes which precede numerical estimation and indeed feed into it (Anobile et al., 2020; Stoianov & Zorzi, 2012; Testolin et al., 2020; Trick & Enns, 1997; Zorzi & Testolin, 2018). Under this latter hypothesis, the psychophysics of estimation in vision could result from constraints inherent to visuospatial memory, and then we would expect that people's behavior in visual tasks not involving number to show equivalent hallmarks to those seen in estimation.

The model presented in Chapter 3 demonstrates that principles of efficient representations can explain many features of number psychophysics, including the discontinuity from exactness to scalar variability. The key idea there was that an efficient encoding of number, using at maximum some number of bits of information, will prioritize representations of small numbers at the expense of large numbers because people tend to need to represent small numbers more frequently (Dehaene & Mehler, 1992; Piantadosi & Cantlon, 2017). That work therefore derived exactness for small numbers (e.g. subitizing) and approximation for large numbers by solving a single, unifying optimization. However, the model did not explain the key step of how numerosities are actually computed from visual input, and therefore does not explain *where* noise in representations of numerosities comes from. Furthermore, that model made the unrealistic assumption that, all else being equal, small and large numerosities are equally easy to perceive—their differing behavioral signatures being solely a matter of frequency of use.

Chapter 4 investigates whether number psychophysics arise from one or more "number systems," as is commonly believed, or if they have their origins in lower-level perceptual processing [2]. We propose an adaptation of the model developed in Chapter 3, which optimally represents objects in

[2] This paper is currently under review. A paper containing an early version of this work won the modeling prize in Perception and Action at the *Cognitive Science Conference*, where it was published in the proceedings as Cheyette et al. (2021), and can be found at https://escholarship.org/uc/item/9hk7s32c.

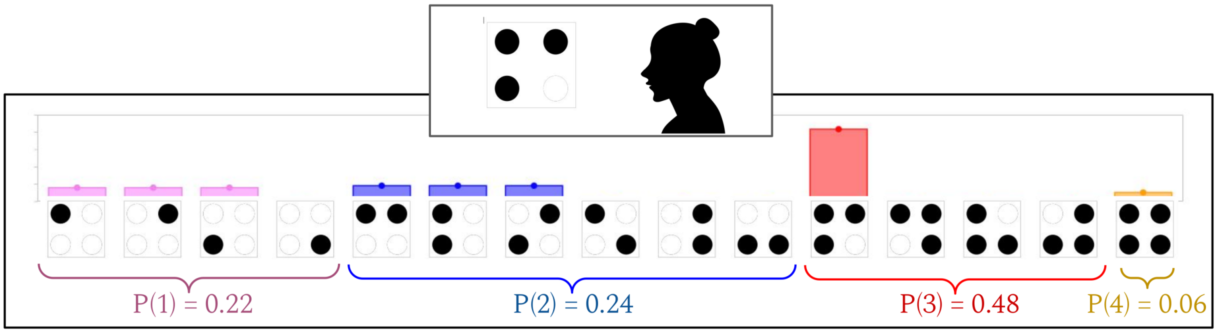

Figure 5: This figure conceptually illustrates how the model works, simplifying it to assume that there are only 4 pixels for clarity. In this example, a person sees a scene with 3 objects, which is represented as a probability distribution over all possibilities of what she saw. Possible arrangements of objects are grouped by numerosity, shown as different colors. To get the probability of a numerosity k, the model simply sums the probability of all possible scenes with numerosity k, highlighted at the bottom.

space (rather than quantities) subject to an information capacity constraint. Quantities are only *implicitly* represented in this model, unlike in the direct optimization of numerosity perception in Chapter 3. However, we show that many of the important psychophysical phenomena associated with numerical cognition — including subitizing and Weber's law — can be derived as downstream consequences.

The model aims to capture how an idealized, information-limited perceptual system would perform if its only aim was to accurately store the presence or absence of objects in various locations. Although this formalizes the idea of object memory—not specifically numerical estimation—its output nonetheless yields psychophysical properties seen in number. Figure 5 illustrates the basic setup, assuming for the sake of clarity that there are only 4 possible object locations (or pixels). When a person sees a particular scene, they encode a probability distribution over each possible arrangement of objects, which is a weighted combination of a prior for small numbers and how well the representation matches their observation (akin to a likelihood). This probability distribution in turn can be converted into a probability distribution over numerosities by summing the probabilities of each scene with a given number of objects. Surprisingly, it turns out that optimizing memory to accurately remember objects' locations results in very similar predictions about number psychophysics as directly optimizing for numerical estimation accuracy. For instance, subitizing, Weber's law, underestimation, and the temporal dynamics of number psychophysics remain qualitatively the same in the two cases. But, importantly, the spatial encoding model additionally predicts that the capacity to remember the locations of objects should significantly influence (if not entirely determine) the capacity to estimate quantities.

We ran two experiments to test whether the model was able to predict both participants' ability to remember the spatial positions of objects *and* their estimates of quantity. The first experiment was a change-localization task to test spatial memory; the other was a numerical estimation task. In both experiments, we manipulated the exposure duration of the displays. We fit the model to human data in both tasks and found that the model was able to capture the key psychophsyics of both tasks, including: how participants' ability to remember objects' locations changed as a function of the number of objects and the exposure duration; how participants' numerical estimates were biased as a function of time; and the cutoff between subitizing and estimation as a function of time. We also found that the inferred capacity limit for remembering objects' spatial locations is nearly identical to the inferred capacity limit for numerical estimation. Moreover, the two capacity limits track very closely over time for numerosities in the range 1-15 and the model closely fits human data in both of these tasks.

In sum, we are able to recover the key properties of numerical cognition in an entirely non-numerical visual task using a visual model; moreover, the patterns of noise and bias in estimation align precisely with the noise inherent to spatial memory, indicating that the psychophysics of number are attributable to perceptual uncertainty rather than number-specific processing. Our results indicate, therefore, that the defining features of numerical cognition can be understood as downstream consequences of basic visual processing, posing a challenge to theories that assume the psychophysics observed in estimation are the result of number-specific processing via one or more "number systems." While there must exist some number-specific processing—quantity must be extracted from visual memory—our findings indicate that Weber's law, subitizing, under-estimation and other effects observed in numerical estimation are not the direct *result* of that processing.

Conclusion

There are two broad themes in this thesis. The first is in finding common functional and mechanistic origins of number psychophysics that have historically been treated as separate phenomena. For instance, people's exact representation of small sets (Burr et al., 2010; Choo & Franconeri, 2014; Feigenson et al., 2004; Jevons, 1871; Revkin et al., 2008) but increasingly imprecise representation of larger sets (Dehaene, 2011; Xu & Spelke, 2000) has been explained as arising from different representational systems (Dehaene, 2011; Feigenson et al., 2004). Our results show that this discontinuity is actually an efficient representation of number given a limited information capacity under a single system. The second broad theme is in demonstrating that numerosity perception cannot be viewed separately from general perceptual processing. We found that visual numerosity perception is strongly tied to visual fixations, which determine the bias and precision of estimates. We also found that the psychophysics of number are largely driven by domain general visual processing — specifically, uncertainty about where objects are in space — and that subitizing, Weber's law, underestimation, and other effects can all be understood as consequences of a limited capacity to represent objects in space. Together, the work in this thesis sheds light on the origins of various puzzling psychophysical phenomena that have been the source of interest and debate for over 150 years (Jevons, 1871).

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